

Extreme Values and Saddle Points

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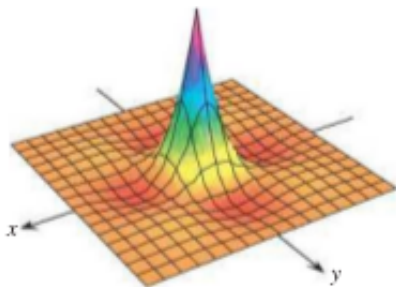
Overview

Continuous functions of two variables assume extreme values on closed, bounded domains. We discuss in the lecture that we can narrow the search for these extreme values by examining the functions' first partial derivatives.

A function of two variables can assume extreme values only at domain boundary points or at interior domain points where both first partial derivatives are zero or where one or both of the first partial derivatives fail to exist.

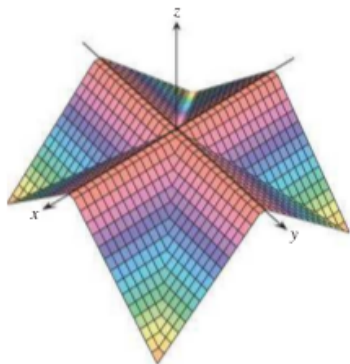
However, the vanishing of derivatives at an interior point (a, b) does not always signal the presence of an extreme value. The surface that is the graph of the function might be shaped like a saddle right above (a, b) and cross its tangent plane there.

Continuous functions of two variables assume extreme values on closed, bounded domains.



The function $z = (\cos x)(\cos y)e^{-\sqrt{x^2+y^2}}$ has a maximum value of 1 and a minimum value of about -0.067 on the square region $|x| \leq 3\pi/2$, $|y| \leq 3\pi/2$.

Continuous functions of two variables assume extreme values on closed, bounded domains.



The **roof surface** $z = \frac{1}{2}(|x| - |y| - |x| - |y|)$ has a maximum value of 0 and a minimum value of $-a$ on the square region $|x| \leq a, |y| \leq a$.

Derivative Tests for Local Extreme Values

To find the local extreme values of a function of a single variable, we look for points where the graph has a **horizontal tangent line**.

At such points, we then look for **local maxima**, **local minima**, and **points of inflection**.

For a function $f(x, y)$ of two variables, we look for points where the surface $z = f(x, y)$ has a horizontal tangent plane.

At such points, we then look for local maxima, local minima, and saddle points.

Maxima and Minima

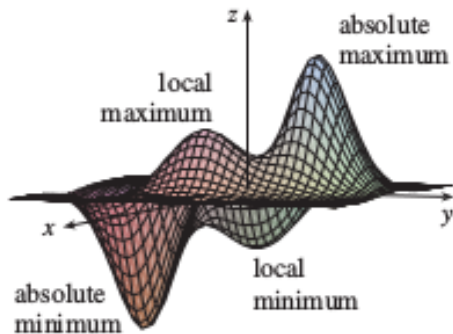
We begin by defining maxima and minima.

Definition 1.

Let $f(x, y)$ be defined on a region R containing the point (a, b) . Then

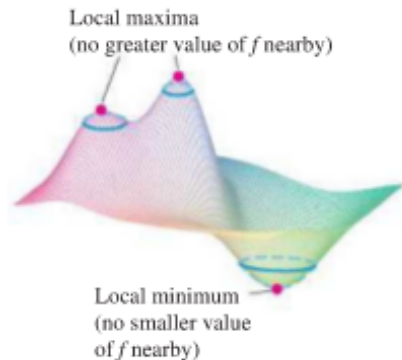
1. $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b) .
2. $f(a, b)$ is a local minimum value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b) .

Maxima and Minima



Maxima and Minima

Local maxima correspond to mountain peaks on the surface $z = f(x, y)$ and local minima correspond to valley bottoms.



At such points the tangent planes, when they exist, are horizontal. Local extrema are also called **relative extrema**.

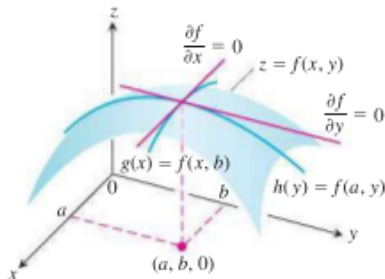
First Derivative Test for Local Extreme Values

As with functions of a single variable, the key to identifying the local extrema is a first derivative test.

Theorem 2 (First Derivative Test for Local Extreme Values).

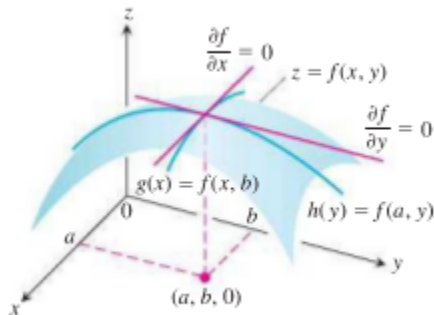
If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$



Proof

If f has a local extremum at (a, b) , then the function $g(x) = f(x, b)$ has a local extremum at $x = a$. Therefore, $g'(a) = 0$.



Now $g'(a) = f_x(a, b)$, so $f_x(a, b) = 0$. A similar argument with the function $h(y) = f(a, y)$ shows that $f_y(a, b) = 0$.

First Derivative Test for Local Extreme Values

If we substitute the values $f_x(a, b) = 0$ and $f_y(a, b) = 0$ into the equation

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0$$

for the tangent plane to the surface $z = f(x, y)$ at (a, b) , the equation reduces to

$$0 \cdot (x - a) + 0 \cdot (y - b) - z + f(a, b) = 0$$

or

$$z = f(a, b).$$

Thus, Theorem (2) says that the surface does indeed have a horizontal tangent plane at a local extremum, provided there is a tangent plane there.

Definition 3.

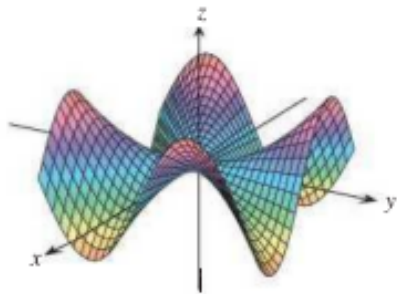
An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a critical point of f .

Theorem (2) says that the only points where a function $f(x, y)$ can assume extreme values are critical points and boundary points.

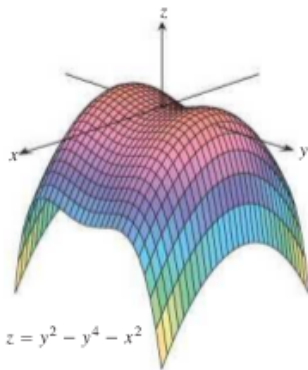
As with differentiable functions of a single variable, not every critical point gives rise to a local extremum. A differentiable function of a single variable might have a point of inflection.

A differentiable function of two variables might have a *saddle point*.

Saddle points at the origin



$$z = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$



$$z = y^2 - y^4 - x^2$$

Saddle Points

Definition 4.

A differentiable function $f(x, y)$ has a saddle point at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$.

The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface.

Example 5.

Find the local extreme values of $f(x, y) = x^2 + y^2 - 4y + 9$.

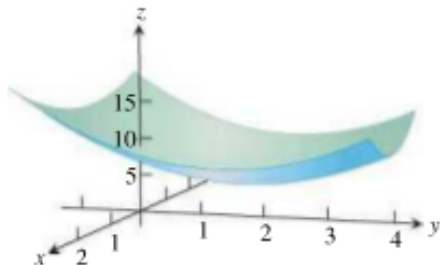
Solution

The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y - 4$ exist everywhere.

Therefore, local extreme values can occur only where

$$f_x = 2x = 0 \quad \text{and} \quad f_y = 2y - 4 = 0.$$

The only possibility is the point $(0, 2)$, where the value of f is 5. Since $f(x, y) = x^2 + (y - 2)^2 + 5$ is never less than 5, we see that the critical point $(0, 2)$ gives a local minimum.



Example

Example 6.

Find the local extreme values (if any) of $f(x, y) = y^2 - x^2$.

Solution : The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = -2x$ and $f_y = 2y$ exist everywhere.

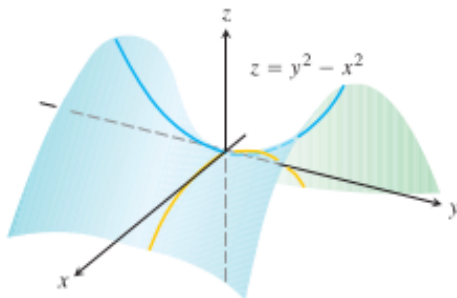
Therefore, local extrema can occur only at the origin $(0,0)$ where $f_x = 0$ and $f_y = 0$.

Along the positive x -axis, however, f has the value $f(x, 0) = -x^2 < 0$;

along the positive y -axis, f has the value $f(0, y) = y^2 > 0$.

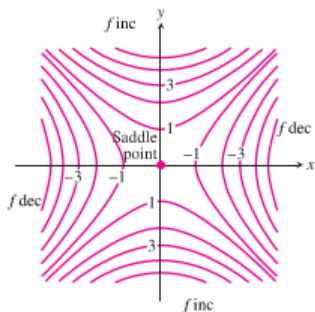
Solution (contd...)

Therefore, every open disk in the xy -plane centered at $(0,0)$ contains points where the function is positive and points where it is negative. The function has a saddle point at the origin and no local extreme values (see the following figure).



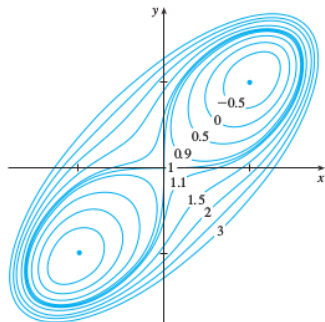
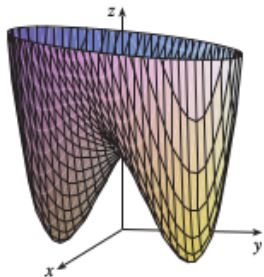
Solution (contd...)

The following figure displays the level curves (they are hyperbolas) of f , and shows the function decreasing and increasing in an alternative fashion among the four groupings of hyperbolas.



Example

A contour map of the function $f = x^4 + y^4 - 4xy + 1$ is shown in the following figure. The level curves near $(1, 1)$ and $(-1, -1)$ are oval in shape and indicate that as we move away from $(1, 1)$ or $(-1, -1)$ in any direction the values of f are increasing. The level curves near $(0, 0)$, on the other hand, resemble hyperbolas. They reveal that as we move away from the origin (where the value of f is 1), the values of f decrease in some directions but increase in other directions. Thus the contour map suggests the presence of the minima and a saddle point.



Remark

That $f_x = f_y = 0$ at an interior point (a,b) of R does not guarantee f has a local extreme value there.

If f and its first and second partial derivatives are continuous on R , however, we may be able to learn more from the following theorem.

Second Derivative Test for Local Extreme Values

Theorem 7 (Second Derivative Test for Local Extreme Values).

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

- i) f has a local maximum at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) .
- ii) f has a local minimum at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) .
- iii) f has a saddle point at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) .
- iv) the test is inconclusive at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b) . In this case, we must find some other way to determine the behavior of f at (a, b) .

Second Derivative Test for Local Extreme Values

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** or **Hessian** of f . It is sometimes easier to remember it in determinant form,

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

Theorem (7) says that if the discriminant is positive at the point (a, b) , then the surface curves the same way **in all directions**: downward if $f_{xx} < 0$, giving rise to a local maximum, and upward if $f_{xx} > 0$, giving a local minimum.

On the other hand, if the discriminant is negative at (a, b) , then the surface curves up in some directions and down in others, so we have a saddle point.

Example

Example 8.

Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution : The function is defined and differentiable for all x and y and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \quad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Solution (contd...)

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \quad f_{yy} = -2, \quad f_{xy} = 1.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \quad \text{and} \quad f_{xx}f_{yy} - f_{xy}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$.

Example

Example 9.

Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Solution : Since f is differentiable everywhere, it can assume extreme values only where

$$f_x = 6y - 6x = 0 \quad \text{and} \quad f_y = 6y - 6y^2 + 6x = 0.$$

From the first of these equations we find $x = y$, and substitution for y into the second equation then gives

$$6x - 6x^2 + 6x = 0 \quad \text{or} \quad 6x(2 - x) = 0.$$

The two critical points are therefore $(0, 0)$ and $(2, 2)$. To classify the critical points, we calculate the second derivatives :

$$f_{xx} = -6, \quad f_{yy} = 6 - 12y, \quad f_{xy} = 6.$$

Solution (contd...)

The discriminant is given by

$$f_{xx}f_{yy} - f_{xy}^2 = (-36 + 72y) - 36 = 72(y - 1).$$

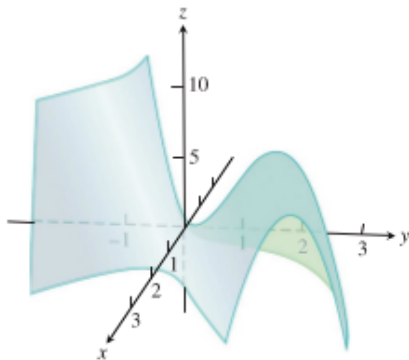
At the critical point $(0, 0)$ we see that the value of the discriminant is the negative number -72 , so the function has a saddle point at the origin.

At the critical point $(2, 2)$ we see that the discriminant has the positive value 72 .

Combining this result with the negative value of the second partial $f_{xx} = -6$, Theorem (7) says that the critical point $(2, 2)$ gives a local maximum value of $f(2, 2) = 12 - 16 - 12 + 24 = 8$.

Solution (contd...)

A graph of the surface is shown in the following figure.



The surface $z = 3y^2 - 2y^3 - 3x^2 + 6xy$ has a saddle point at the origin and a local maximum at the point $(2, 2)$.

Absolute Maxima and Minima on Closed Bounded Regions

We organize the search for the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region R into three steps.

1. *List the interior points of R where f may have local maxima and minima and evaluate f at these points. These are the critical points of f .*
2. *List the boundary points of R where f has local maxima and minima and evaluate f at these points. We show how to do this shortly.*
3. *Look through the lists for the maximum and minimum values of f . These will be the absolute maximum and minimum values of f on R . Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of f appear somewhere in the lists made in Steps 1 and 2.*

Example

Example 10.

Find the absolute maximum and minimum values of

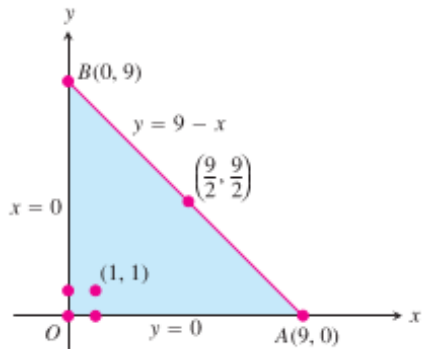
$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, $y = 9 - x$.

Solution : Since f is differentiable, the only places where f can assume these values are points inside the triangle where $f_x = f_y = 0$ and points on the boundary.

Solution (contd...)

This triangular region is the domain of the function



(a) Interior points : For these we have

$$f_x = 2 - 2x = 0, \quad f_y = 2 - 2y = 0,$$

yielding the single point $(x, y) = (1, 1)$. The values of f there is

$$f(1, 1) = 4.$$

Solution (contd...)

(b) Boundary points : We take the triangle one side at a time:

(i) On the segment OA , $y = 0$. The function

$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

may now be regarded as a function of x defined on the closed interval $0 \leq x \leq 9$. Its extreme values may occur at the endpoints

$$x = 0 \quad \text{where} \quad f(0, 0) = 2$$

$$x = 9 \quad \text{where} \quad f(9, 0) = 2 + 18 - 81 = -61$$

and at interior points where $f'(x, 0) = 2 - 2x = 0$. The only interior point where $f'(x, 0) = 0$ is $x = 1$, where

$$f(x, 0) = f(1, 0) = 3.$$

Solution (contd...)

(ii) On the segment OB , $x = 0$ and

$$f(x, y) = f(0, y) = 2 + 2y - y^2.$$

We know from the symmetry of f in x and y and from the analysis we just carried out that the candidates on this segment are

$$f(0, 0) = 2, \quad f(0, 9) = -61, \quad f(0, 1) = 3.$$

(iii) We have already accounted for the values of f at the endpoints of AB , so we need only look at the interior points of AB . With $y = 9 - x$, we have

$$f(x, y) = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2 = -61 + 18x - 2x^2.$$

Solution (contd...)

Setting $f'(x, 9 - x) = 18 - 4x = 0$ gives

$$x = \frac{18}{4} = \frac{9}{2}.$$

At this value of x ,

$$y = 9 - \frac{9}{2} = \frac{9}{2} \quad \text{and} \quad f(x, y) = f\left(\frac{9}{2}, \frac{9}{2}\right) = -\frac{41}{2}.$$

Summary : We list all the candidates : $4, 2, -61, 3, -(41/2)$.

The maximum is 4, which f assumes at $(1, 1)$.

The minimum is -61 , which f assumes at $(0, 9)$ and $(9, 0)$.

Example

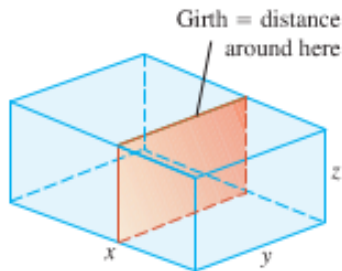
Solving extreme value problems with algebraic constraints on the variables usually requires the method of Lagrange multipliers introduced in the next section. But sometimes we can solve such problems directly, as in the next example.

Example 11.

A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 in. Find the dimensions of an acceptable box of largest volume.

Solution : Let $x, y,$ and z represent the length, width, and height of the rectangular box, respectively. Then the girth is $2y + 2z$. We want to maximize the volume $V = xyz$ of the box satisfying $x + 2y + 2z = 108$ (the largest box is accepted by the delivery company).

Solution (contd...)



Thus, we can write the volume of the box as a function of two variables:

$$\begin{aligned} V(y, z) &= (108 - 2y - 2z)yz && V = xyz \text{ and} \\ & && x = 108 - 2y - 2z \\ &= 108yz - 2y^2z - 2yz^2. \end{aligned}$$

Solution (contd...)

Setting the first partial derivatives equal to zero,

$$V_y(y, z) = 108z - 4yz - 2z^2 = (108 - 4y - 2z)z = 0$$

$$V_z(y, z) = 108y - 2y^2 - 4yz = (108 - 2y - 4z)y = 0,$$

gives the critical points $(0, 0)$, $(0, 54)$, $(54, 0)$, and $(18, 18)$. The volume is zero at $(0, 0)$, $(0, 54)$, $(54, 0)$, which are not maximum values. At the point $(18, 18)$, we apply the Second Derivative Test :

$$V_{yy} = -4z, \quad V_{zz} = -4y, \quad V_{yz} = 108 - 4y - 4z.$$

Then

$$V_{yy}V_{zz} - V_{yz}^2 = 16yz - 16(27 - y - z)^2.$$

Solution (contd...)

Thus,

$$V_{yy}(18, 18) = -4(18) < 0$$

and

$$[V_{yy}V_{6zz} - V_{yz}^2]_{(18,18)} = 16(18)(18) - 16(-9)^2 > 0$$

imply that $(18, 18)$ gives a maximum volume.

The dimensions of the package are $x = 108 - 2(18) - 2(18) = 36$ in., $y = 18$ in., and $z = 18$ in.

The maximum volume is $V = (36)(18)(18) = 11,664$ in³, or 6.75 ft³.

Summary of Max-Min Tests

Despite the power of Theorem (7), we urge you to remember its limitations. It does not apply to boundary points of a function's domain, where it is possible for a function to have extreme values along with nonzero derivatives. Also, it does not apply to points where either f_x or f_y fails to exist.

The extreme values of $f(x, y)$ can occur only at

- i) **boundary points** of the domain of f
- ii) **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fails to exist).

Summary of Max-Min Tests

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

- i) $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \implies$ **local maximum**
- ii) $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \implies$ **local minimum**
- iii) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a, b) \implies$ **saddle point**
- iv) $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a, b) \implies$ **test is inconclusive**

Extreme Values on Parametrized Curves

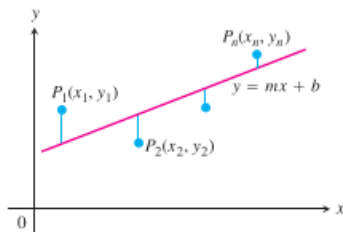
To find the extreme values use of a function $f(x, y)$ on a curve $x = x(t), y = y(t)$, we treat f as a function of the single variable t and use the Chain Rule to find where df/dt is zero. As in any other single-variable case, the extreme values of f are then found among the values at the

- (a) critical points (points where df/dt is zero or fails to exist), and
- (b) endpoints of the parameter domain.

Least squares and regression lines

When we try to fit a line $y = mx + b$ to a set of numerical data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, we usually choose the line that minimizes the sum of the squares of the vertical distances from the points to the line. In theory, this means finding the values of m and b that minimize the value of the function

$$w = (mx_1 + b - y_1)^2 + \dots + (mx_n + b - y_n)^2. \quad (1)$$



To fit a line to noncollinear points, we choose the line that minimizes the sum of the squares of the deviations.

Least squares and regression lines

One can show that the values of m and b that do this are

$$m = \frac{(\sum x_k)(\sum y_k) - n \sum x_k y_k}{(\sum x_k)^2 - n \sum x_k^2},$$
$$b = \frac{1}{n} \left(\sum y_k - m \sum x_k \right),$$

with all sums running from $k = 1$ to $k = n$.

Many scientific calculators have these formulas built in, enabling you to find m and b with only a few keystrokes after you have entered the data.

Least squares and regression lines

The line $y = mx + b$ determined by these values of m and b is called the **least squares line**, **regression line**, or **trend line** for the data under study. Finding a least squares line lets you

1. summarize data with a simple expression,
2. predict values of y for other, experimentally untried values of x ,
3. handle data analytically.

Exercise 12.

Find all the local maxima, local minima, and saddle points of the following functions.

1. $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$

2. $f(x, y) = 2xy - x^2 - 2y^2 + 3x + 4$

3. $f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$

4. $f(x, y) = x^2 - y^2 - 2x + 4y + 6$

5. $f(x, y) = x^2 + 2xy$

6. $f(x, y) = \sqrt{56x^2 - 8y^2 - 16x - 31} + 1 - 8x$

7. $f(x, y) = x^3 + 3xy^2 - 15x + y^3 - 15y$

8. $f(x, y) = \frac{1}{x^2 + y^2 - 1}$

9. $f(x, y) = e^y - ye^x$

10. $f(x, y) = 2 \ln x + \ln y - 4x - y$

Solution for (1.), (2.) and (3.) in Exercise 12

- $f_x(x, y) = 2y - 10x + 4 = 0$ and $f_y(x, y) = 2x - 4y + 4 = 0 \Rightarrow x = \frac{2}{3}$ and $y = \frac{4}{3} \Rightarrow$ critical point is $(\frac{2}{3}, \frac{4}{3})$; $f_{xx}(\frac{2}{3}, \frac{4}{3}) = -10$, $f_{yy}(\frac{2}{3}, \frac{4}{3}) = -4$, $f_{xy}(\frac{2}{3}, \frac{4}{3}) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(\frac{2}{3}, \frac{4}{3}) = 0$
- $f_x(x, y) = 2y - 2x + 3 = 0$ and $f_y(x, y) = 2x - 4y = 0 \Rightarrow x = 3$ and $y = \frac{3}{2} \Rightarrow$ critical point is $(3, \frac{3}{2})$; $f_{xx}(3, \frac{3}{2}) = -2$, $f_{yy}(3, \frac{3}{2}) = -4$, $f_{xy}(3, \frac{3}{2}) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(3, \frac{3}{2}) = \frac{17}{2}$
- $f_x(x, y) = 2x - 2y - 2 = 0$ and $f_y(x, y) = -2x + 4y + 2 = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow$ critical point is $(1, 0)$; $f_{xx}(1, 0) = 2$, $f_{yy}(1, 0) = 4$, $f_{xy}(1, 0) = -2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(1, 0) = 0$

Solution for (4.), (5.) and (6.) in Exercise 12

4. $f_x(x, y) = 2x - 2 = 0$ and $f_y(x, y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow$ critical point is $(1, 2)$; $f_{xx}(1, 2) = 2$, $f_{yy}(1, 2) = -2$, $f_{xy}(1, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point
5. $f_x(x, y) = 2x + 2y = 0$ and $f_y(x, y) = 2x = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow$ critical point is $(0, 0)$; $f_{xx} = 2$, $f_{yy}(0, 0) = 0$, $f_{xy}(0, 0) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point
6. $f_x(x, y) = \frac{112x - 8x}{\sqrt{56x^2 - 8y^2 - 16x - 31}} - 8 = 0$ and $f_y(x, y) = \frac{-8y}{\sqrt{56x^2 - 8y^2 - 16x - 31}} = 0 \Rightarrow$ critical point is $(\frac{16}{7}, 0)$; $f_{xx}(\frac{16}{7}, 0) = -\frac{8}{15}$, $f_{yy}(\frac{16}{7}, 0) = -\frac{8}{15}$, $f_{xy}(\frac{16}{7}, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = \frac{64}{225} > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(\frac{16}{7}, 0) = -\frac{16}{7}$

Solution for (7.) in Exercise 12

7. $f_x(x, y) = 3x^2 + 3y^2 - 15 = 0$ and $f_y(x, y) = 6xy + 3y^2 - 15 = 0 \Rightarrow$
critical point are $(2, 1)$, $(-2, -1)$, $(0, \sqrt{5})$, and $(0, -\sqrt{5})$; for $(2, 1)$; $f_{xx}(2, 1) = 6x|_{(2,1)} =$
 12 , $f_{yy}(2, 1) = (6x + 6y)|_{(2,1)} = 18$, $f_{xy}(2, 1) = 6y|_{(2,1)} = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 180 >$
 0 and $f_{xx} > 0 \Rightarrow$ local minimum of $f(2, 1) = -30$; for $(-2, -1)$: $f_{xx}(-2, -1) =$
 $6x|_{(-2,-1)} = -12$, $f_{yy}(-2, -1) = (6x + 6y)|_{(-2,-1)} = -18$, $f_{xy}(-2, -1) = 6y|_{(-2,-1)} =$
 $-6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 180 > 0$ and $f_{xx} < 0 \Rightarrow$ local minimum of $f(-2, -1) =$
 30 ; for $(0, \sqrt{5})$: $f_{xx}(0, \sqrt{5}) = 6x|_{(0,\sqrt{5})} = 0$, $f_{yy}(0, \sqrt{5}) = (6x + 6y)|_{(0,\sqrt{5})} =$
 $6\sqrt{5}$, $f_{xy}(0, \sqrt{5}) = 6y|_{(0,\sqrt{5})} = 6\sqrt{5} \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -180 < 0 \Rightarrow$ saddle point; for $(0,$

$$\text{for } (0, -\sqrt{5}) : f_{xx}(0, -\sqrt{5}) = 6x|_{(0,-\sqrt{5})} = 0, f_{yy}(0, -\sqrt{5}) = (6x + 6y)|_{(0,-\sqrt{5})} = -6\sqrt{5},$$

$$f_{xy}(0, -\sqrt{5}) = 6y|_{(0,-\sqrt{5})} = -6\sqrt{5} \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -180 < 0 \Rightarrow \text{saddle point}$$

Solution for (8.), (9.) and (10.) in Exercise 12

8. $f_x(x, y) = \frac{-2x}{(x^2+y^2-1)^2} = 0$ and $f_y(x, y) = \frac{-2y}{(x^2+y^2-1)^2} = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow$ the critical point is $(0, 0)$; $f_{xx} = \frac{4x^2-2y^2+2}{(x^2+y^2-1)^2}$, $f_{yy} = \frac{-2x^2+4y^2+2}{(x^2+y^2-1)^2}$, $f_{xy} = \frac{8xy}{(x^2+y^2-1)^2}$; $f_{xx}(0, 0) = -2$, $f_{yy}(0, 0) = -2$, $f_{xy}(0, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(0, 0) = -1$
9. $f_x(x, y) = -ye^x = 0$ and $f_y(x, y) = e^y - e^x = 0 \Rightarrow$ critical point is $(0, 0)$; $f_{xx}(2, 0) = 0$, $f_{xy}(2, 0) = -1$, $f_{yy}(2, 0) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point
10. $f_x(x, y) = -4 + \frac{2}{x} = 0$ and $f_y(x, y) = -1 + \frac{1}{y} = 0 \Rightarrow$ critical point is $(\frac{1}{2}, 1)$; $f_{xx}(\frac{1}{2}, 1) = -8$, $f_{yy}(\frac{1}{2}, 1) = -1$, $f_{xy}(\frac{1}{2}, 1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 8 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(\frac{1}{2}, 1) = -3 - 2 \ln 2$

Finding Absolute Extrema

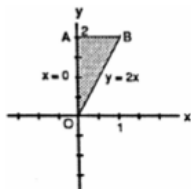
Exercise 13.

Find the absolute maxima and minima of the functions of the given domains.

- 1. $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant.*
- 2. $D(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 4$, $y = x$.*
- 3. $T(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \leq x \leq 5$, $-3 \leq y \leq 0$.*

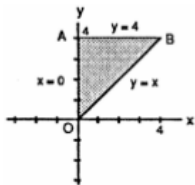
Solution for (1.) in Exercise 13

- (i) On OA, $f(x, y) = f(0, y) = y^2 - 4y + 1$ on $0 \leq y \leq 2$; $f'(0, y) = 2y - 4 = 0 \Rightarrow y = 2$; $f(0, 0) = 1$ and $f(0, 2) = -3$
- (ii) On AB, $f(x, y) = f(x, 2) = 2x^2 - 4x - 3$ on $0 \leq x \leq 1$; $f'(x, 2) = 4x - 4 = 0 \Rightarrow x = 1$; $f(0, 2) = -3$ and $f(1, 2) = -5$
- (iii) On OB, $f(x, y) = f(x, 2x) = 6x^2 - 12x + 1$ on $0 \leq x \leq 1$; endpoint values have been found above; $f'(x, 2x) = 12x - 12 = 0 \Rightarrow x = 1$ and $y = 2$, but $(1, 2)$ is not an interior point of OB
- (iv) For interior point of the triangular region, $f_x(x, y) = 4x - 4 = 0$ and $f_y(x, y) = 2y - 4 = 0 \Rightarrow x = 1$ and $y = 2$, but $(1, 2)$ is not an interior point of the region. Therefore, the absolute maximum is 1 at $(0, 0)$ and the absolute minimum is -5 at $(1, 2)$.



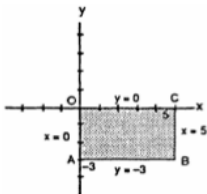
Solution for (2.) in Exercise 13

- (i) On OA , $D(x, y) = D(0, y) = y^2 + 1$ on $0 \leq y \leq 4$; $D'(0, y) = 2y = 0 \Rightarrow y = 0$; $D(0, 0) = 1$ and $D(0, 4) = 17$
- (ii) On AB , $D(x, y) = D(x, 4) = x^2 - 4x + 17$ on $0 \leq x \leq 4$; $D'(x, 4) = 2x - 4 = 0 \Rightarrow x = 2$ and $(2, 4)$ is an interior point of AB ; $D(2, 4) = 13$ and $D(4, 4) = D(0, 4) = 17$
- (iii) On OB , $D(x, y) = D(x, x) = x^2 + 1$ on $0 \leq x \leq 4$; $D'(x, x) = 2x = 0 \Rightarrow x = 0$ and $y = 0$, which is not an interior point of OB ; endpoint values have been found above
- (iv) For interior points of the triangular region, $f_x(x, y) = 2x - y = 0$ and $f_y(x, y) = -x + 2y = 0 \Rightarrow x = 0$ and $y = 0$, which is not an interior point of the region. Therefore, the absolute maximum is 17 at $(0, 4)$ and $(4, 4)$, and the absolute minimum is 1 at $(0, 0)$.



Solution for (3.) in Exercise 13

- (i) on OC , $T(x, y) = T(x, 0) = x^2 - 6x + 2$ on $0 \leq x \leq 5$; $T'(x, 0) = 2x - 6 = 0 \Rightarrow x = 3$ and $y = 0$; $T(3, 0) = -7$, $T(0, 0) = 2$, and $T(5, 0) = -3$
- (ii) On CB , $T(x, y) = T(5, y) = y^2 + 5y - 3$ on $-3 \leq y \leq 0$; $T'(5, y) = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$ and $x = 5$; $T(5, -\frac{5}{2}) = -\frac{37}{4}$ and $T(5, -3) = -9$
- (iii) On AB , $T(x, y) = T(x, -3) = x^2 - 9x + 11$ on $0 \leq x \leq 5$; $T'(x, -3) = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$ and $y = -3$; $T(\frac{9}{2}, -3) = -\frac{37}{4}$ and $T(0, -3) = 11$
- (iv) On AO , $T(x, y) = T(0, y) = y^2 + 2$ on $-3 \leq y \leq 0$; $T'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$, but $(0, 0)$ is not an interior point of AO .
- (v) For interior points of the rectangular region, $T_x(x, y) = 2x + y - 6 = 0$ and $T_y(x, y) = x + 2y = 0 \Rightarrow x = 4$ and $y = -2$, an interior critical point with $T(4, -2) = -10$. Therefore the absolute maximum is 11 at $(0, -3)$ and the absolute minimum is -10 at $(4, -2)$.

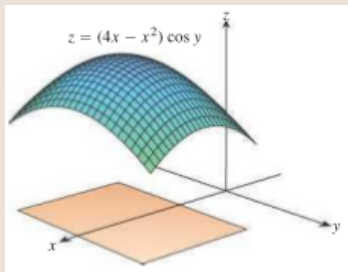


Finding Absolute Extrema

Exercise 14.

Find the absolute maxima and minima of the functions of the given domains.

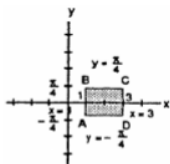
1. $f(x, y) = (4x - x^2) \cos y$ on the rectangular plate $1 \leq x \leq 3$, $-\pi/4 \leq y \leq \pi/4$ (see accompanying figure).



2. $f(x, y) = 4x - 8xy + 2y + 1$ on the triangular plate bounded by the lines $x = 0$, $y = 0$, $x + y = 1$ in the first quadrant.

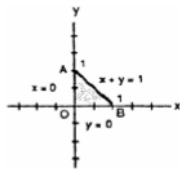
Solution for (1.) in Exercise 14

- (i) On AB , $f(x, y) = f(1, y) = 3 \cos y$ on $-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$; $f'(1, y) = -3 \sin y = 0 \Rightarrow y = 0$ and $x = 1$; $f(1, 0) = 3$, $f(1, -\frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$, and $f(1, \frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$.
- (ii) On CD , $f(x, y) = f(3, y) = 3 \cos y$ on $-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$; $f'(3, y) = -3 \sin y = 0 \Rightarrow y = 0$ and $x = 3$; $f(3, 0) = 3$, $f(3, -\frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$ and $f(3, \frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$
- (iii) On BC , $f(x, y) = f(x, \frac{\pi}{4}) = \frac{\sqrt{2}}{2}(4x - x^2)$ on $1 \leq x \leq 3$; $f'(x, \frac{\pi}{4}) = \sqrt{2}(2 - x) = 0 \Rightarrow x = 2$ and $y = \frac{\pi}{4}$; $f(2, \frac{\pi}{4}) = 2\sqrt{2}$, $f(1, \frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$, and $f(3, \frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$
- (iv) On AD , $f(x, y) = f(x, -\frac{\pi}{4}) = \frac{\sqrt{2}}{2}(4x - x^2)$ on $1 \leq x \leq 3$; $f'(x, -\frac{\pi}{4}) = \sqrt{2}(2 - x) = 0 \Rightarrow x = 2$ and $y = -\frac{\pi}{4}$; $f(2, -\frac{\pi}{4}) = 2\sqrt{2}$, $f(1, -\frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$, and $f(3, -\frac{\pi}{4}) = \frac{3\sqrt{2}}{2}$
- (v) For interior point of the region,
 $f_x(x, y) = (4 - 2x) \cos y = 0$ and $f_y(x, y) = -(4x - x^2) \sin y = 0 \Rightarrow x = 2$ and $y = 0$,
which is an interior critical point with $f(2, 0) = 4$. Therefore the absolute maximum is 4
at $(2, 0)$ and the absolute minimum is $\frac{3\sqrt{2}}{2}$ at $(3, -\frac{\pi}{4})$, $(3, \frac{\pi}{4})$, $(1, -\frac{\pi}{4})$, and $(1, \frac{\pi}{4})$.



Solution for (2.) in Exercise 14

- (i) On OA , $f(x, y) = f(0, y) = 2y + 1$ on $0 \leq y \leq 1$; $f'(0, y) = 2 \Rightarrow$ no interior critical points, $f(0, 0) = 1$ and $f(0, 1) = 3$
- (ii) On OB , $f(x, y) = f(x, 0) = 4x + 1$ on $0 \leq x \leq 1$; $f'(x, 0) = 4 \Rightarrow$ no interior critical points; $f(1, 0) = 5$
- (iii) On AB , $f(x, y) = f(x, -x + 1) = 8x^2 - 6x + 3$ on $0 \leq x \leq 1$; $f'(x, -x + 1) = 16x - 6 = 0 \Rightarrow x = \frac{3}{8}$ and $y = \frac{5}{8}$; $f(\frac{3}{8}, \frac{5}{8}) = \frac{15}{8}$, $f(0, 1) = 3$, and $f(1, 0) = 5$
- (iv) For interior points of the triangular region, $f_x(x, y) = 4 - 8y = 0$ and $f_y(x, y) = -8x + 2 = 0 \Rightarrow y = \frac{1}{2}$ and $x = \frac{1}{4}$ which is an interior critical point with $f(\frac{1}{4}, \frac{1}{2}) = 2$. Therefore the absolute maximum is 5 at $(1, 0)$ and the absolute minimum is 1 at $(0, 0)$.



Finding Absolute Extrema

Exercise 15.

Find the absolute maxima and minima of the functions of the given domains.

1. Find two numbers a and b with $a \leq b$ such that

$$\int_a^b (6 - x - x^2) dx$$

has its largest value.

2. Find two numbers a and b with $a \leq b$ such that

$$\int_a^b (24 - 2x - x^2)^{1/3} dx$$

has its largest value.

Solution for Exercise 15

1. Let $F(a, b) = \int_a^0 (6 - x - x^2) dx$ where $a \leq b$. The boundary of the domain of F is the line $a = b$ in the ab -plane, and $F(a, a) = 0$, so F is identically 0 on the boundary of its domain. For interior critical points we have:
 $\frac{\delta F}{\delta a} = -(6 - a - a^2) = 0 \Rightarrow a = -3, 2$ and $\frac{\delta f}{\delta b} = (6 - b - b^2) = 0 \Rightarrow b = -3, 2$. Since $a \leq b$, there is only one interior critical point $(-3, 2)$ and $f(-3, 2) = \int_{-3}^2 (6 - x - x^2) dx$ gives the area under the parabola $y = 6 - x - x^2$ that is above the x -axis. Therefore, $a = -3$ and $b = 2$.
2. Let $F(a, b) = \int_a^0 (24 - 2x - x^2)^{1/3} dx$ where $a \leq b$. The boundary of the domain of F is the line $a = b$ and on this line F is identically 0. For interior critical points we have:
 $\frac{\delta F}{\delta a} = -(24 - 2a - a^2)^{1/3} = 0 \Rightarrow a = 4, -6$ and
 $\frac{\delta f}{\delta b} = (24 - 2b - b^2)^{1/3} = 0 \Rightarrow b = 4, -6$. Since $a \leq b$, there is only one critical point $(-6, 4)$ and $F(-6, 4) = \int_{-6}^4 (24 - 2x - x^2) dx$ gives the area under the curve $y = (24 - 2x - x^2)^{1/3}$ that is above the x -axis. Therefore, $a = -6$ and $b = 4$.

Exercise 16.

1. Temperatures : A flat circular plate has the shape of the region $x^2 + y^2 \leq 1$. The plate, including the boundary where $x^2 + y^2 = 1$, is heated so that the temperature at the point (x, y) is

$$T(x, y) = x^2 + 2y^2 - x.$$

Find the temperatures at the hottest and coldest points on the plate.

2. Find the critical point of

$$f(x, y) = xy + 2x - \ln x^2y$$

in the open first quadrant ($x > 0, y > 0$) and show that f takes on a minimum there.

Solution for Exercise 16

1. $T_x(x, y) = 2x - 1 = 0$ and $T_y(x, y) = 4y = 0 \Rightarrow x = \frac{1}{2}$ and $y = 0$ with $T(\frac{1}{2}, 0) = -\frac{1}{4}$; on the boundary $x^2 + y^2 = 1$: $T(x, y) = -x^2 - x + 2$ for $-1 \leq x \leq 1 \Rightarrow T'(x, y) = -2x - 1 = 0 \Rightarrow x = -\frac{1}{2}$ and $y = \pm \frac{\sqrt{3}}{2}$; $T(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = \frac{9}{4}$, $T(-\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{9}{4}$, $T(-1, 0) = 2$, and $T(1, 0) = 0 \Rightarrow$ the hottest is $2\frac{1}{4}^\circ$ at $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$; the coldest is $-\frac{1}{4}^\circ$ at $(\frac{1}{2}, 0)$.
2. $f_x(x, y) = y + 2 - \frac{2}{x}$ and $f_y(x, y) = x - \frac{1}{y} = 0 \Rightarrow x = \frac{1}{2}$ and $y = 2$; $f_{xx}(\frac{1}{2}, 2) = \frac{2}{x^2}|_{(\frac{1}{2}, 2)} = 8$, $f_{yy}(\frac{1}{2}, 2) = \frac{1}{y^2} = \frac{1}{y^2}|_{(\frac{1}{2}, 2)} = \frac{1}{4}$, $f_{xy}(\frac{1}{2}, 2) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 1 > 0$ and $f_{xx} > 0 \Rightarrow$ a local minimum of $f(\frac{1}{2}, 2) = 2 - \ln \frac{1}{2} = 2 + \ln 2$

Exercise 17.

Find the maxima, minima, and saddle points of $f(x, y)$, if any, given that

(a) $f_x = 2x - 4y$ and $f_y = 2y - 4x$

(b) $f_x = 2x - 2$ and $f_y = 2y - 4$

(c) $f_x = 9x^2 - 9$ and $f_y = 2y + 4$

Describe your reasoning in each case.

Solution for Exercise 17

- (a) $f_x(x, y) = 2x - 4y = 0$ and $f_y(x, y) = 2y - 4x = 0 \Rightarrow x = 0$ and $y = 0$; $f_{xx}(0, 0) = 2$, $f_{yy}(0, 0) = 2$, $f_{xy}(0, 0) = -4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -12 < 0 \Rightarrow$ saddle point at $(0, 0)$
- (b) $f_x(x, y) = 2x - 2 = 0$ and $f_y(x, y) = 2y - 4 = 0 \Rightarrow x = 1$ and $y = 2$; $f_{xx}(1, 2) = 2$, $f_{yy}(1, 2) = 2$, $f_{xy}(1, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum at $(1, 2)$
- (c) $f_x(x, y) = 9x^2 - 9$ and $f_y(x, y) = 2y + 4 = 0 \Rightarrow x = \pm 1$ and $y = -2$; $f_{xx}(1, -2) = 18x|_{(1, -2)} = 18$, $f_{yy}(1, -2) = 2$, $f_{xy}(1, -2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum at $(1, -2)$; $f_{xx}(-1, -2) = -18$, $f_{yy}(-1, -2) = 2$, $f_{xy}(-1, -2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point at $(-1, -2)$

Exercise 18.

The discriminant $f_{xx}f_{yy} - f_{xy}^2$ is zero at the origin for each of the following functions, so the Second Derivative Test fails there. Determine whether the function has a maximum, a minimum, or neither at the origin by imagining what the surface $z = f(x, y)$ looks like. Describe your reasoning in each case.

(a) $f(x, y) = x^2y^2$

(b) $f(x, y) = 1 - x^2y^2$

(c) $f(x, y) = xy^2$

(d) $f(x, y) = x^3y^2$

(e) $f(x, y) = x^3y^3$

(f) $f(x, y) = x^4y^4$

Solution for Exercise 18

- (a) Minimum at $(0, 0)$ since $f(x, y) > 0$ for all other (x, y)
- (b) Maximum of 1 at $(0, 0)$ since $f(x, y) < 1$ for all other (x, y)
- (c) Neither since $f(x, y) < 0$ for $x < 0$ and $f(x, y) > 0$ for $x > 0$
- (d) Neither since $f(x, y) < 0$ for $x < 0$ and $f(x, y) > 0$ for $x > 0$
- (e) Neither since $f(x, y) < 0$ for $x < 0$ and $y > 0$, but $f(x, y) > 0$ for $x > 0$ and $y > 0$
- (f) Minimum at $(0, 0)$ since $f(x, y) > 0$ for all other (x, y)

Exercise 19.

1. Show that $(0, 0)$ is a critical point of $f(x, y) = x^2 + kxy + y^2$ no matter what value the constant k has. (Hint : Consider two cases: $k = 0$ and $k \neq 0$.)
2. For what values of the constant k does the Second Derivative Test guarantee that $f(x, y) = x^2 + kxy + y^2$ will have a saddle point at $(0, 0)$? A local minimum at $(0, 0)$? For what values of k is the Second Derivative Test inconclusive? Give reasons for your answers.
3. If $f_x(a, b) = f_y(a, b) = 0$, must f have a local maximum or minimum value at (a, b) ? Give reasons for your answer.

Solution for Exercise 19

1. If $k = 0$, then $f(x, y) = x^2 + y^2 \Rightarrow f_x(x, y) = 2x = 0$ and $f_y(x, y) = 2y = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow (0, 0)$ is the only critical point. If $k \neq 0$, $f_x(x, y) = 2x + ky = 0 \Rightarrow y = -\frac{2}{k}x$; $f_y(x, y) = kx + 2y = 0 \Rightarrow kx + 2(-\frac{2}{k}x) = 0 \Rightarrow kx - \frac{4x}{k} = 0 \Rightarrow (k - \frac{4}{k})x = 0 \Rightarrow x = 0$ or $k = \pm 2 \Rightarrow y = (-\frac{2}{k})(0) = 0$ or $y = \pm x$; in any case $(0, 0)$ is a critical point.
2. (see the Exercise above): $f_{xx}(x, y) = 2$, $f_{yy}(x, y) = 2$, and $f_{xy}(x, y) = k \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 - k^2$; f will have a saddle point at $(0, 0)$ if $4 - k^2 < 0 \Rightarrow k > 2$ or $k < -2$; f will have a local minimum at $(0, 0)$ if $4 - k^2 > 0 \Rightarrow -2 < k < 2$; the test is inconclusive if $4 - k^2 = 0 \Rightarrow k = \pm 2$.
3. No; for example $f(x, y) = xy$ has a saddle point at $(a, b) = (0, 0)$ where $f_x = f_y = 0$.

Exercise 20.

1. *Can you conclude anything about $f(a, b)$ if f and its first and second partial derivatives are continuous throughout a disk centered at the critical point (a, b) and $f_{xx}(a, b)$ and $f_{yy}(a, b)$ differ in sign? Give reasons for your answer.*
2. *Among all the points on the graph of $z = 10 - x^2 - y^2$ that lie above the plane $x + 2y + 3z = 0$, find the point farthest from the plane.*
3. *Find the point on the graph of $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$.*
4. *Find three positive numbers whose sum is 3 and whose product is a maximum.*

Solution for Exercise 20

1. If $f_{xx}(a, b)$ and $f_{yy}(a, b)$ differ in sign, then $f_{xx}(a, b)f_{yy}(a, b) < 0$ so $f_{xx}f_{yy} - f_{xy}^2 < 0$. The surface must therefore have a saddle point at (a, b) by the second derivative test.
2. We want the point on $z = 10 - x^2 - y^2$ where the tangent plane is parallel to the plane $x + 2y + 3z = 0$. To find a normal vector to $z = 10 - x^2 - y^2$ let $w = z + x^2 + y^2 - 10$. Then $\nabla w = 2xi + 2yj + k$ is normal to $z = 10 - x^2 - y^2$ at (x, y) . The vector ∇w is parallel to $i + 2j + 3k$ which is normal to the plane $x + 2y + 3z = 0$ if $6xi + 6yj + 3k = i + 2j + 3k$ or $x = \frac{1}{6}$ and $y = \frac{1}{3}$. Thus the point is $(\frac{1}{6}, \frac{1}{3}, 10 - \frac{1}{36}, -\frac{1}{9})$ or $(\frac{1}{6}, \frac{1}{3}, \frac{355}{36})$.
3. We want the point on $z = x^2 + y^2 + 10$ where the tangent plane is parallel to the plane $x + 2y - z = 0$. Let $w = z - x^2 - y^2 - 10$, then $\nabla w = -2xi - 2yj + k$ is normal to $z = x^2 + y^2 + 10$ at (x, y) . The vector ∇w is parallel to $i + 2j - k$ which is normal to the plane if $x = \frac{1}{2}$ and $y = 1$. Thus the point $(\frac{1}{2}, 1, \frac{1}{4} + 1 + 10)$ or $(\frac{1}{2}, 1, \frac{45}{4})$ is the point on the surface $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$.
4. $p(x, y, z) = xyz; x + y + z = 3 \Rightarrow z = 3 - x - y \Rightarrow p(x, y) = xy(3 - x - y) = 3xy - x^2y - xy^2 \Rightarrow P_x(x, y) = 3y - 2xy - y^2 = 0$ and $p_y(x, y) = 3x - x^2 - 2xy = 0 \Rightarrow$ critical points are $(0, 0), (0, 3), (3, 0)$, and $(1, 1)$; for $(0, 0) \Rightarrow z = 3; p_{xx}(0, 0) = 0, p_{yy}(0, 0) = 0, p_{xy}(0, 0) = 3 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = -9 < 0 \Rightarrow$ saddle point; for $(0, 3) \Rightarrow z = 0; p_{xx}(0, 3) = -6, p_{yy}(0, 3) = 0, p_{xy}(0, 3) = -3 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = -9 < 0 \Rightarrow$ saddle point; for $(3, 0) \Rightarrow z = 0; p_{xx}(3, 0) = 0, p_{yy}(3, 0) = -6, p_{xy}(3, 0) = -3 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = -9 < 0 \Rightarrow$ saddle point; for $(1, 1) \Rightarrow z = 1; p_{xx}(1, 1) = -2, p_{yy}(1, 1) = -2, p_{xy}(1, 1) = -1 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = 3 > 0$ and $p_{xx} < 0 \Rightarrow$ local maximum of $p(1, 1, 1) = 1$

Exercise 21.

1. Find the dimensions of the rectangular box of maximum volume that can be inscribed inside the sphere $x^2 + y^2 + z^2 = 4$.
2. Consider the function $f(x, y) = x^2 + y^2 + 2xy - x - y + 1$ over the square $0 \leq x \leq 1$ and $0 \leq y \leq 1$.
 - (a) show that f has an absolute minimum along the line segment $2x + 2y = 1$ in this square. What is the absolute minimum value?
 - (b) Find the absolute maximum value of f over the square.

Solution for (1.) in Exercise 21

1. $V(x, y, z) = (2x)(2y)(2z) = 8xyz$; $x^2 + y^2 + z^2 = 4 \Rightarrow z = \sqrt{4 - x^2 - y^2} \Rightarrow V(x, y) = 8xy\sqrt{4 - x^2 - y^2}$, $x \geq 0$ and $y \geq 0 \Rightarrow V_x(x, y) = \frac{32y - 16x^2y - 8y^3}{\sqrt{4 - x^2 - y^2}} = 0$ and $V_y(x, y) = \frac{32x - 16xy^2 - 8x^3}{\sqrt{4 - x^2 - y^2}} = 0 \Rightarrow$ critical points are $(0, 0)$, $(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$, $(\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$, $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$, and $(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}})$. Only $(0, 0)$ and $(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$ are in the domain $x \geq 0$ and $y \geq 0$. $V(0, 0) = 0$ and $V(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}) = \frac{64}{3\sqrt{3}}$; on $x = 0$, $0 \leq y \leq 2 \Rightarrow V(0, y) = 8(0)y\sqrt{4 - 0^2 - y^2} = 0$, no critical points, $V(0, 0) = 0$, $V(0, 2) = 0$; on $y = 0$, $0 \leq x \leq 2 \Rightarrow V(x, 0) = 8x(0)\sqrt{4 - x^2 - 0^2} = 0$, no critical points, $V(0, 0) = 0$, $V(2, 0) = 0$; on $y = \sqrt{4 - x^2}$, $0 \leq x \leq 2 \Rightarrow V(x - \sqrt{4 - x^2}) = 8x\sqrt{4 - x^2}\sqrt{4 - x^2 - (\sqrt{4 - x^2})^2} = 0$ no critical points, $V(0, 2) = 0$, $V(2, 0) = 0$. Thus, there is a maximum volume of $\frac{64}{3\sqrt{3}}$ if the box is $\frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}}$.

Solution for (2.) in Exercise 21

2. (a) (i) On $x = 0$, $f(x, y) = f(0, y) = y^2 - y + 1$ for $0 \leq y \leq 1$;
 $f'(0, y) = 2y - 1 = 0 \Rightarrow y = \frac{1}{2}$ and $x = 0$; $f(0, \frac{1}{2}) = \frac{3}{4}$, $f(0, 0) = 1$,
and $f(0, 1) = 1$
- (ii) on $y = 1$, $f(x, y) = f(x, 1) = x^2 + x + 1$ for $0 \leq x \leq 1$;
 $f'(x, 1) = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$ and $y = 1$, but $(-\frac{1}{2}, 1)$ is outside the
domain; $f(0, 1) = 1$ and $f(1, 1) = 3$
- (iii) On $x = 1$, $f(x, y) = f(1, y) = y^2 + y + 1$ for $0 \leq y \leq 1$;
 $f'(1, y) = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$ and $x = 1$, but $(1, -\frac{1}{2})$ is outside the
domain; $f(1, 0) = 1$ and $f(1, 1) = 3$
- (iv) On $y = 0$, $f(x, y) = f(x, 0) = x^2 - x + 1$ for $0 \leq x \leq 1$;
 $f'(x, 0) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ and $y = 0$; $f(\frac{1}{2}, 0) = \frac{3}{4}$; $f(0, 0) = 1$,
and $f(1, 0) = 1$
- (v) On the interior of the square, $f_x(x, y) = 2x + 2y - 1 = 0$ and
 $f_y(x, y) = 2y + 2x - 1 = 0 \Rightarrow 2x + 2y = 1 \Rightarrow (x + y) = \frac{1}{2}$. Then
 $f(x, y) = x^2 + y^2 + 2xy - x - y + 1 = (x + y)^2 - (x + y) + 1 = \frac{3}{4}$ is
the absolute minimum value when $2x + 2y = 1$.
- (b) The absolute maximum is $f(1, 1) = 3$.

Exercise 22.

Find the absolute maximum and minimum values of the following functions on the given curves.

Functions :

(a) $f(x, y) = 2x + 3y$

(b) $g(x, y) = xy$

(c) $h(x, y) = x^2 + 3y^2$

Curves :

(i) *The semiellipse* $(x^2/9) + (y^2/4) = 1, y \geq 0$

(ii) *The quarter ellipse* $(x^2/9) + (y^2/4) = 1, x \geq 0, y \geq 0$

Use the parametric equations $x = 3 \cos t, y = 2 \sin t$.

Solution for (a.) in Exercise 22

$$(a) \quad \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 2 \frac{dx}{dt} + 3 \frac{dy}{dt} = -6 \sin t + 6 \cos t = 0 \Rightarrow \sin t = \cos t \Rightarrow t = \frac{\pi}{4} \text{ for } 0 \leq t \leq \pi.$$

(i) On the semi-ellipse, $\frac{x^2}{9} + \frac{y^2}{4} = 1, y \geq 0$,

$f(x, y) = 2x + 3y = 6 \cos t + 6 \sin t = 6 \left(\frac{\sqrt{2}}{2} \right) + 6 \left(\frac{\sqrt{2}}{2} \right) = 6\sqrt{2}$ at $t = \frac{\pi}{4}$. At the endpoints, $f(-3, 0) = -6$ and $f(3, 0) = 6$. The absolute minimum is $f(-3, 0) = -6$ when $t = \pi$; the absolute maximum is $f\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 6\sqrt{2}$ when $t = \frac{\pi}{4}$.

(ii) On the quarter ellipse, at the endpoints $f(0, 2) = 6$ and $f(3, 0) = 6$. The absolute minimum is $f(3, 0) = 6$ and $f(0, 2) = 6$ when $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $f\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 6\sqrt{2}$ when $t = \frac{\pi}{4}$.

Solution for (b.) in Exercise 22

$$(b) \quad \frac{dg}{dt} = \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial g}{\partial y} \frac{dy}{dt} = y \frac{dx}{dt} + x \frac{dy}{dt} = (2 \sin t)(-3 \sin t) + (3 \cos t)(2 \cos t) = 6(\cos^2 t - \sin^2 t) = 6 \cos 2t = 0 \Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4} \text{ for } 0 \leq t \leq \pi.$$

- (i) On the semi-ellipse, $g(x, y) = xy = 6 \sin t \cos t$. Then $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$, and $g\left(-\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = -3$ when $t = \frac{3\pi}{4}$. At the endpoints, $g(-3, 0) = g(3, 0) = 0$. The absolute minimum is $g\left(-\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = -3$ when $t = \frac{3\pi}{4}$; the absolute maximum is $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$.
- (ii) On the quarter ellipse, at the endpoints $g(0, 2) = 0$ and $g(3, 0) = 0$. The absolute minimum is $g(3, 0) = 0$ and $g(0, 2) = 0$ at $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$.

Solution for (c.) in Exercise 22

- (c) $\frac{dh}{dt} = \frac{\partial h}{\partial x} \frac{dx}{dt} + \frac{\partial h}{\partial y} \frac{dy}{dt} = 2x \frac{dx}{dt} + 6y \frac{dy}{dt} =$
 $(6 \cos t)(-3 \sin t) + (12 \sin t)(2 \cos t) = 6 \sin t \cos t = 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi$
for $0 \leq t \leq \pi$, yielding the points $(3, 0)$, $(0, 2)$, and $(-3, 0)$.
- (i) On the semi-ellipse, $y \geq 0$ so that $h(3, 0) = 9$, $h(0, 2) = 12$, and $h(-3, 0) = 9$. The absolute minimum is $h(3, 0) = 9$ and $h(-3, 0) = 9$ when $t = 0, \pi$ respectively; the absolute maximum is $h(0, 2) = 12$ when $t = \frac{\pi}{2}$.
- (ii) On the quarter ellipse, the absolute minimum is $h(3, 0) = 9$ when $t = 0$; the absolute maximum is $h(0, 2) = 12$ when $t = \frac{\pi}{2}$.

Exercise 23.

Find the absolute maximum and minimum values of the following functions on the given curves.

Functions :

(a) $f(x, y) = x^2 + y^2$

(b) $g(x, y) = 1/(x^2 + y^2)$

Curves :

(i) *The line $x = t, y = 2 - 2t$*

(ii) *The line segment $x = t, y = 2 - 2t, 0 \leq t \leq 1$*

Solution for (a.) in Exercise 23

$$(a) \quad \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

- (i) $x = t$ and $y = 2 - 2t \Rightarrow \frac{df}{dt} = (2t)(1) + 2(2 - 2t)(-2) = 10t - 8 = 0 \Rightarrow t = \frac{4}{5} \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$ with $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{16}{25} + \frac{4}{25} = \frac{4}{5}$. The absolute minimum is $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$ when $t = \frac{4}{5}$; there is no absolute maximum along the line.
- (ii) For the endpoints: $t = 0 \Rightarrow x = 0$ and $y = 2$ with $f(0, 2) = 4$; $t = 1 \Rightarrow x = 1$ and $y = 0$ with $f(1, 0) = 1$. The absolute minimum is $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$ at the interior critical point when $t = \frac{4}{5}$; the absolute maximum is $f(0, 2) = 4$ at the endpoint when $t = 0$.

Solution for (b.) in Exercise 23

$$(b) \quad \frac{dg}{dt} = \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial g}{\partial y} \frac{dy}{dt} = \left[\frac{-2x}{(x^2+y^2)^2} \right] \frac{dx}{dt} + \left[\frac{-2y}{(x^2+y^2)^2} \right] \frac{dy}{dt}$$

- (i) $x = t$ and $y = 2 - 2t \Rightarrow x^2 + y^2 = 5t^2 - 8t + 4 \Rightarrow \frac{dg}{dt} = -(5t^2 - 8t + 4)^{-2} [(-2t)(1) + (-2)(2 - 2t)(-2)] = -(5t^2 - 8t + 4)^{-2} (-10t + 8) = 0 \Rightarrow t = \frac{4}{5} \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$ with $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{1}{\left(\frac{4}{5}\right)^2} = \frac{5}{4}$. The absolute maximum is $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{5}{4}$ when $t = \frac{4}{5}$; there is no absolute minimum along the line since x and y can be as large as we please.
- (ii) For the endpoints: $t = 0 \Rightarrow x = 0$ and $y = 2$ with $g(0, 2) = \frac{1}{4}$; $t = 1 \Rightarrow x = 1$ and $y = 0$ with $g(1, 0) = 1$. The absolute minimum is $g(0, 2) = \frac{1}{4}$ when $t = 0$; the absolute maximum is $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{5}{4}$ when $t = \frac{4}{5}$.

Exercise 24.

Find the least squares line for each set of data points. Then use the linear equation you obtain to predict the value of y that would correspond to $x = 4$.

1. $(-2, 0), (0, 2), (2, 3)$
2. $(-1, 2), (0, 1), (3, -4)$

Solution for Exercise 24

1. $m = \frac{(0)(5) - 3(6)}{(0)^2 - 3(8)} = \frac{3}{4}$ and $b = \frac{1}{3} \left[5 - \frac{3}{4}(0) \right] = \frac{5}{3} \Rightarrow y = \frac{3}{4}x + \frac{5}{3}; y|_{x=4} = \frac{14}{3}$

k	x_k	y_k	x_k^2	$x_k y_k$
1	-2	0	4	0
2	0	2	0	0
3	2	3	4	6
Σ	0	5	8	6

2. $m = \frac{(2)(-1) - 3(-14)}{(2)^2 - 3(10)} = -\frac{20}{13}$ and
 $b = \frac{1}{3} \left[-1 - \left(-\frac{20}{3}\right)(2) \right] = \frac{9}{13} \Rightarrow y = -\frac{20}{13}x + \frac{9}{13}; y|_{x=4} = -\frac{71}{13}$

k	x_k	y_k	x_k^2	$x_k y_k$
1	-1	2	1	-2
2	0	1	0	0
3	3	-4	9	-12
Σ	2	-1	10	-14

Question 25.

The function $f(x, y) = x^2 + 4y^2 - 2x + 8y - 1$ for $(x, y) \in \mathbb{R}^2$ has

- (A) local minimum at $(1, -1)$
- (B) a saddle point at $(1, -1)$
- (C) local maximum at $(1, -1)$
- (D) None of the other options.

The correct answer is
local minimum at $(1, -1)$.

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